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# WAVE FRONT SETS OF SOLUTIONS TO ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA

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## 1. Introduction

We consider elastic wave propagation problems in plane-stratified media  $\mathbf{R}^3$  with the planer interface  $x_3 = 0$ . This problem is formulated as an elastic mixed or initial-interfece problem in a stratified media.

An elastic equation has two speeds. Pressure or Primary wave (for simplicity called P wave) and Share or Secondary wave (S wave). P wave is a longitudinal wave and S wave is a transversal wave. In general the speed of P wave is grater than that of S wave. In plane-stratified media problem, a lower half-space  $\mathbf{R}_-^3$  called *Medium I* has  $P_1$  and  $S_1$  waves and an upper half-space  $\mathbf{R}_+^3$  called *Medium II* has  $P_2$  and  $S_2$  waves. The speed of  $P_1$  (resp.  $P_2$ ) wave is grater than that of  $S_1$  (resp.  $S_2$ ) wave. So the order relation of the speeds of  $P_1$ ,  $P_2$ ,  $S_1$ , and  $S_2$  waves are six cases. Here we assume  $P_2, S_2, P_1, S_1$  waves in order of speed since it is the most complex case.

We put unit impulse Dirac's delta in the lower half-space Medium I. Then  $P_1$  incident wave which speed is faster than  $S_1$  incident wave bumps against the interface and causes  $P_1$  and  $S_1$  reflected waves in Medium I and  $P_2$  and  $S_2$  refracted waves in Medium II as in Figure 1.

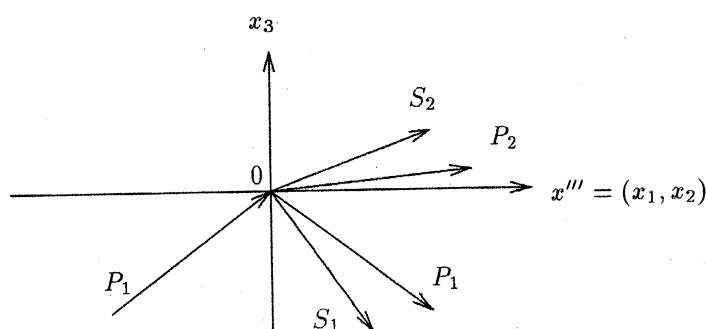


Figure 1 Reflected waves and refracted waves

Moreover when time goes on, lateral waves in other words glancing waves or total reflected (or refracted) waves arise. In Figure 2, dotted arrows show  $P_2$ - $P_1$  and  $P_2$ - $S_1$  lateral waves in Medium I, and  $P_2$ - $S_2$  lateral wave in Medium II for

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$P_1$  incident wave.  $P_2$ - $P_1$  lateral wave means that the wave originally should have been  $P_2$  reflected wave tends to total reflection, then becomes source and causes  $P_1$  reflected wave.

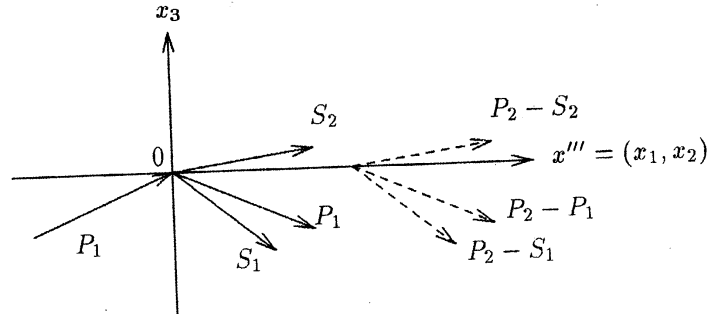


Figure 2 Lateral waves

We have 11th kind of lateral waves in all. It is a characteristic of our elastic wave propagation problems in stratified media. If half-space problem which has two speeds, there exist only one kind of lateral wave. If plane-stratified media problem that each medium has one speed, there exist only one kind of lateral wave. Thus this elastic wave propagation problems in plane-stratified media has many lateral waves.

In this paper we prove the above physical situation mathematically by using a expression of an inner estimate of singularities. The main technical tool of our analysis is a localization method.

We gave an inner estimate of the location of singularities of the reflected and refracted Riemann functions by making use of the localization method [7,8]. This method is first studied by M. F. Atiyah, R. Bott, and L. Gårding [1] and L. Hörmander [2] for initial value problem, then studied by M. Matsumura [5], M. Tsuji [9], and S. Wakabayashi [10,11] for half-space mixed problem.

In this paper, we give an outer estimate of wave front sets of the incident, reflected and refracted Riemann functions by making use of the localization method. Atiyah-Bott-Gårding [1] studied the outer estimate of wave front sets of solutions to initial value problem. Wakabayashi studied for half-space mixed problem [12], and for more general case [13]. We analysis an outer estimate of wave front sets of the Riemann functions to the elastic mixed problem based on Wakabayashi's theorem [12, Theorem 4.2]. Combining the inner estimate and the outer estimate, we obtain the exact wave front sets of the elastic mixed problem in stratified media.

I would like to express my gratitude to Professor Seiichiro Wakabayashi for his invaluable suggestions.

## 2. Formulation of Problems

We consider elastic wave propagation problems in the following plane-stratified media  $\mathbf{R}^3$  with the planar interface  $x_3 = 0$ :

$$(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1) & \text{for } x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2) & \text{for } x_3 > 0. \end{cases}$$

Here the constants  $\lambda_1, \lambda_2, \mu_1, \mu_2$  are called the Lamé constants and the constants

$\rho_1, \rho_2$  are densities. We shall denote the lower half-space  $\mathbf{R}_-^3$  by *Medium I* and the upper half-space  $\mathbf{R}_+^3$  by *Medium II*, respectively, as in Figure 3.

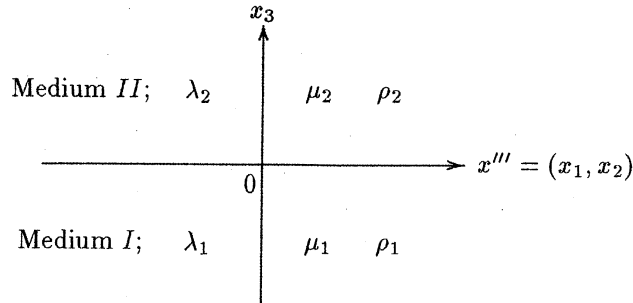


Figure 3 Stratified media

We assume that

$$(1.1) \quad \lambda_i + \mu_i > 0, \quad \mu_i > 0, \quad \rho_i > 0, \quad i = 1, 2.$$

(1.1) is the natural assumption in practical situation. From the roots of the characteristic equations of  $P^I(D)$  and  $P^{II}(D)$  which are defined below  $3 \times 3$  matrix valued hyperbolic partial differential operators in Medium I and Medium II, respectively, we obtain two speeds correspond to Pressure or Primary wave (for simplicity called P wave) and Share or Secondary wave (S wave) on each medium. P wave is a longitudinal wave and S wave is a transversal wave.  $c_{p1}$  denotes the speed of P wave in Medium I and  $c_{s1}$  denotes the speed of S wave in Medium I.  $c_{p2}$  and  $c_{s2}$  denote the speed of P and S wave in Medium II, respectively. They are given by

$$c_{pi}^2 = \frac{\lambda_i + 2\mu_i}{\rho_i}, \quad c_{si}^2 = \frac{\mu_i}{\rho_i}, \quad i = 1, 2.$$

By assumption (1.1), the speed of P wave is greater than that of S wave in each medium. On account of this, these are six cases of the order relation of the speeds of  $\{c_{p1}, c_{s1}, c_{p2}, c_{s2}\}$ . Here we assume that

$$c_{s1} < c_{p1} \leq c_{s2} < c_{p2},$$

since if we put an unit impulse Dirac's delta in Medium I, it is the case that the most number of lateral waves are appeared. The other cases can be treated in a similar manner (cf. [6, Section 3]).

Let  $x = (x_0, x_1, x_2, x_3) = (x', x_3) = (x_0, x'') = (x_0, x''', x_3)$  in  $\mathbf{R}^4$ . The variable  $x_0$  will play a role of time, and  $x'' = (x_1, x_2, x_3)$  will play that of space.  $\xi$  is a real dual variable of  $x$  and is equal to  $(\xi_0, \xi_1, \xi_2, \xi_3) = (\xi', \xi_3) = (\xi_0, \xi'') = (\xi_0, \xi''', \xi_3)$  in  $\mathbf{R}_\xi^4$ . We use the differential symbol  $D_j = i^{-1} \partial / \partial x_j$  ( $j = 0, 1, 2, 3$ ), where  $i = \sqrt{-1}$ . We shall denote by  $\mathbf{R}_-^n$  the half-space  $\{x = (x_1, \dots, x_n) \in \mathbf{R}^n \mid x_n < 0\}$  and by  $\mathbf{R}_+^n$  the half-space  $\{x = (x_1, \dots, x_n) \in \mathbf{R}^n \mid x_n > 0\}$ , and also use the notation  $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ .

Let  $u(x) = {}^t(u_1(x), u_2(x), u_3(x)) \in \mathbf{R}^3$  be the displacement vector at time  $x_0$  and position  $x''$ . The propagation problems of elastic waves in the stratified media

is formulated as mixed (initial-interface value) problem:

$$\begin{cases} P^I(D)u(x) = f(x), & x_0 > 0, x'' = (x_1, x_2, x_3) \in \mathbf{R}_-^3, \\ P^{II}(D)u(x) = f(x), & x_0 > 0, x'' = (x_1, x_2, x_3) \in \mathbf{R}_+^3, \\ u(x)|_{x_3=-0} = u(x)|_{x_3=+0}, & x_0 > 0, x''' \in \mathbf{R}^2, \\ B^I(D)u(x)|_{x_3=-0} = B^{II}(D)u(x)|_{x_3=+0}, & x_0 > 0, x''' \in \mathbf{R}^2, \\ D_0^k u(x)|_{x_0=0} = g_k(x''), & k = 0, 1, x'' \in \mathbf{R}^3. \end{cases}$$

Here

$$P^I(D)u = -D_0^2 Eu + \frac{\lambda_1 + \mu_1}{\rho_1} \nabla_{x''} (\nabla_{x''} \cdot u) + \frac{\mu_1}{\rho_1} \Delta_{x''} u,$$

is a  $3 \times 3$  matrix valued second order hyperbolic differential operator with constant coefficients where  $E$  is a  $3 \times 3$  identity matrix,

$$(B^I(D)u)_k = i\lambda_1 (\nabla_{x''} \cdot u) \delta_{k3} + 2\mu_1 \varepsilon_{k3}(u), \quad k = 1, 2, 3,$$

are the  $k$ -th component of symmetric stress tensors of  $B^I(D)u$  where

$$\varepsilon_{k3}(u) = i/2 (D_3 u_k + D_k u_3), \quad k = 1, 2, 3,$$

are strain tensors. The  $P^{II}(D)u$  and  $B^{II}(D)u$  are defined by replacing  $\lambda_1, \mu_1, \rho_1$  by  $\lambda_2, \mu_2, \rho_2$ , respectively.

If we put unit impulse Dirac's delta  $\delta(x - y)$  with  $x_0 \geq y_0$  and  $y_3 < 0$ , that is, put it in Medium I, then the Riemann function of this elastic mixed problem is given by the following:

$$G(x, y) = \begin{cases} E^I(x - y) - F^I(x, y) & \text{for } x_3 < 0, \\ F^{II}(x, y) & \text{for } x_3 > 0. \end{cases}$$

We call  $E^I(x - y)$ ,  $F^I(x, y)$ , and  $F^{II}(x, y)$  the incident, reflected, and refracted Riemann functions, respectively, because these are corresponding to incident, reflected, and refracted waves, respectively.  $E^I(x)$  is the fundamental solution in Medium I describing an incident wave defined by

$$E^I(x) = (2\pi)^{-4} \int_{\mathbf{R}_\xi^4} e^{ix \cdot (\xi + i\eta)} P^I(\xi + i\eta)^{-1} d\xi, \quad \eta \in -\gamma_0 \vartheta - \Gamma(\det P^I, \vartheta),$$

where  $\gamma_0$  is a positive real number,  $\vartheta$  and  $\Gamma(\det P^I, \vartheta)$  are defined Definition 1.4 below, and  $P^I(\xi + i\eta)^{-1}$  is a  $3 \times 3$  inverse matrix. Taking partial Fourier-Laplace transform with respect to  $x'$  for the mixed problem, we obtain a interface value problem for ordinary differential equation with parameters. Then taking partial inverse Fourier-Laplace transform for the solution, we obtain explicit expressions of reflected and refracted Riemann functions  $F^I(x, y)$  and  $F^{II}(x, y)$ .

We define a wave front set  $WF(u)$  and a analytic wave front set  $WF_A(u)$  (cf. [3], [4], [11]).

**Definition 1.1.** Let  $u(x) \in \mathcal{D}'(X)$ . Then the wave front set  $WF(u)$  is defined as the complement in  $X \times (\mathbf{R}^n \setminus \{0\})$  of the collection of the points  $(x^0, \xi^0)$  such that there exist a conic neighborhood  $\Delta$  of  $\xi^0$  in  $\mathbf{R}^n \setminus \{0\}$  and  $\phi \in C_0^\infty(X)$  such that  $\phi(x^0) \neq 0$  and

$$|\mathcal{F}[\phi u](\xi)| \leq C_N(1 + |\xi|)^{-N} \quad \text{when } \xi \in \Delta \text{ and } N = 0, 1, 2, \dots$$

Here  $\mathcal{F}$  denotes the Fourier transform.

For the definition of a analytic wave front set  $WF_A(u)$ , we prepare that there exist a bounded sequence  $\{\phi_N\}$  in  $C_0^\infty$  such that  $\phi_N = 1$  on a fixed neighborhood of  $x^{0'}$  in  $X$ , independent of  $N$ , and

$$|D^\alpha \phi_N| \leq C(CN)^{|\alpha|} \quad \text{for } |\alpha| \leq N.$$

**Definition 1.2.** Let  $u(x) \in \mathcal{D}'(X)$ . Then the analytic wave front set  $WF_A(u)$  is defined as the complement in  $X \times (\mathbf{R}^n \setminus \{0\})$  of the collection of the points  $(x^0, \xi^0)$  such that for some sequence  $\{\phi_N\}$  of the above type there exists a conic neighborhood  $\Delta$  of  $\xi^0$  in  $\mathbf{R}^n \setminus \{0\}$  with

$$|\mathcal{F}[\phi_N u](\xi)| \leq C(CN)^N(1 + |\xi|)^{-N} \quad \text{when } \xi \in \Delta, \quad N = 0, 1, 2, \dots$$

By Definition 1.1 and Definition 1.2, we obtain

$$WF(u) \subset WF_A(u),$$

and  $WF(u)$  and  $WF_A(u)$  are closed subsets of  $X \times (\mathbf{R}^n \setminus \{0\})$ .

We define a localization of polynomials according to Atiyah-Bott-Gårding (cf. [1]):

**Definition 1.3.** Let  $P(\xi)$  be a polynomial of degree  $m \geq 0$  and develop  $\nu^m P(\nu^{-1}\xi + \eta)$  in ascending power of  $\nu$ :

$$\nu^m P(\nu^{-1}\xi + \eta) = \nu^p P_\xi(\eta) + O(\nu^{p+1}) \quad \text{as } \nu \rightarrow 0,$$

where  $P_\xi(\eta)$  is the first coefficient that does not vanish identically in  $\eta$ . The polynomial  $P_\xi(\eta)$  is the localization of  $P$  at  $\xi$ , the number  $p$  is the multiplicity of  $\xi$  relative to  $P$ .

Moreover we introduce the following:

**Definition 1.4.**  $\Gamma = \Gamma(P, \vartheta)$  is the component of  $\mathbf{R}_\eta^n \setminus \{\eta \in \mathbf{R}_\eta^n, P(\eta) = 0\}$  which contains  $\vartheta = (1, 0, \dots, 0) \in \mathbf{R}^n$ . Moreover  $\Gamma' = \Gamma'(P, \vartheta) = \{x \in \mathbf{R}^n \mid x \cdot \eta \geq 0, \eta \in \Gamma\}$  is the dual cone of  $\Gamma$  and is called the propagation cone.

### 3. Results

We obtain the exact wave front sets of the elastic mixed problem in stratified media by combining an inner estimate and an outer estimate. First we mention about the results of the incident Riemann function, namely fundamental solution of Medium I  $E^I(x)$ . This proposition is a version of the theorem proved by Atiyah-Bott-Gårding [1, Theorem 4.10] adopted the present context.

**Proposition.** For  $\xi^0 \in \mathbf{R}_\xi^4 \setminus \{0\}$  satisfying  $(\det P_j^I)(\xi^0) = 0$  ( $j \in \{p_1, s_1\}$ ), that is,

$$(\det P_{p_1}^I)(\xi^0) = \xi_0^{02} - c_{p_1}^2 |\xi^{0''}|^2 = 0,$$

or

$$(\det P_{s_1}^I)(\xi^0) = \xi_0^{02} - c_{s_1}^2 |\xi^{0''}|^2 = 0,$$

then we have

$$\lim_{\nu \rightarrow \infty} \nu e^{-i\nu x \cdot \xi^0} E^I(x) = E_{j\xi^0}^I(x), \quad j \in \{p_1, s_1\},$$

in the distribution sense with respect to  $x \in \mathbf{R}^4$ , where

$$E_{j\xi^0}^I(x) = (2\pi)^{-4} \int_{\mathbf{R}_\xi^4} e^{ix \cdot (\xi + i\eta)} \frac{(\text{cof } P^I)_{\xi^0}(\xi + i\eta)}{(\det P_j^I)_{\xi^0}(\xi + i\eta)} d\xi$$

for  $\eta \in -\vartheta - \Gamma(\det P_j^I, \vartheta)$  and  $j \in \{p_1, s_1\}$

with a positive real  $s$  large enough. Moreover we have

$$WF(E^I(x)) = \bigcup_{\xi^0 \neq 0} (\text{supp } E_{p_1\xi^0}^I(x) \cup \text{supp } E_{s_1\xi^0}^I(x)) \times \{\xi^0\},$$

and

$$\text{supp } E_{j\xi^0}^I(x) = (\Gamma_{j\xi^0})' = \left\{ x \in \mathbf{R}^4 : x \cdot \eta \geq 0 \text{ for any } \eta \in \Gamma_{j\xi^0} \right.$$

$$\left. u = \Gamma((\det P_j^I)_{\xi^0}(\eta), \vartheta) \right\}, \quad \vartheta = (1, 0, 0, 0), \quad j \in \{p_1, s_1\}.$$

In general,  $\text{supp } E_{j\xi^0}^I(x) \subset (\Gamma_{j\xi^0})'$  ( $j = \{p_1, s_1\}$ ), more precisely  $ch[\text{supp } E_{j\xi^0}^I(x - y)] = (\Gamma_{j\xi^0})'$ , where  $ch$  denotes convex hull. However in our problem we obtain  $\text{supp } E_{j\xi^0}^I(x) = (\Gamma_{j\xi^0})'$ .

Secondly we mention about main result. Since we take a partial Fourier-Laplace transform with respect to  $x'$  of  $\delta(x - y)$  regarding  $y'$  as a parameter,  $y'$  appears only in the form  $x' - y'$ . So we put

$$\tilde{F}^\iota(x, y_3) = F^\iota(x, 0, y_3), \quad \iota = \{I, II\}.$$

The following Main Theorem shows the exact wave front sets of  $\tilde{F}^I(x, y_3)$  and  $\tilde{F}^{II}(x, y_3)$ .

**Main Theorem.** For  $\xi^0 \in \mathbf{R}_\xi^4 \setminus \{0\}$  satisfying  $(\det P_j^I)(\xi^0) = 0$  ( $j \in \{p_1, s_1\}$ ) we have the following:

(1) For the reflected Riemann function  $\tilde{F}^I(x, y_3)$ , we have

$$(3.1) \quad \lim_{\nu \rightarrow \infty} \nu e^{-i\nu \{x' \cdot \xi^{0'} + x_3 \tau_k^-(\xi^{0'}) - y_3 \xi_3^0\}} \tilde{F}^I(x, y_3) = \tilde{F}_{j\xi^0 k}^I(x, y_3),$$

$$(j, k) = \{(p_1, p_1), (p_1, s_1), (s_1, p_1), (s_1, s_1)\}$$

and if  $\xi^{0'}$  are zeros of  $\tau_m^+(\zeta')$ , that is,  $\xi^{0'}$  satisfy  $|\xi^{0'''}| = \xi_0^0/c_m$  ( $m \in \{p_1, p_2, s_2\}$ ), then we have

(3.2)

$$\lim_{\nu \rightarrow \infty} \left\{ \nu^{\frac{3}{2}} e^{-i\nu\{x' \cdot \xi^{0'} + x_3 \tau_k^-(\xi^{0'}) - y_3 \xi_3^0\}} \tilde{F}^I(x, y_3) - \nu^{\frac{1}{2}} \tilde{F}_{j\xi^0 k}^I(x, y_3) \right\} = \tilde{F}_{j\xi^0 km}^I(x, y_3),$$

$$(j, k, m) = \{(p_1, p_1, p_2), (p_1, p_1, s_2), (p_1, s_1, p_2), (p_1, s_1, s_2), (s_1, p_1, p_2),$$

$$(s_1, p_1, s_2), (s_1, s_1, p_2), (s_1, s_1, s_2), (s_1, s_1, p_1)\}$$

in the distribution sense with respect to  $(x, y_3) \in \mathbf{R}_-^4 \times \mathbf{R}_-$ .

Moreover we have

$$WF(\tilde{F}^I(x', x_3, y_3)) = WF_A(\tilde{F}^I(x', x_3, y_3)) = \bigcup_{\xi^0 \neq 0} \left[ \bigcup_{\substack{(j,k)=\{(p_1,p_1),(p_1,s_1),\\(s_1,p_1),(s_1,s_1)\}}} \left( \text{supp } \tilde{F}_{j\xi^0 k}^I(x', x_3, y_3) \times \{(\xi^{0'}, \tau_k^-(\xi^{0'}), -\xi_3^0)\} \right) \right. \\ \left. \bigcup_{\substack{(j,k,m)=\\ \{(p_1,p_1,p_2),(p_1,p_1,s_2),(p_1,s_1,p_2),\\(p_1,s_1,s_2),(s_1,p_1,p_2),(s_1,p_1,s_2),\\(s_1,s_1,p_2),(s_1,s_1,s_2),(s_1,s_1,p_1)\}}} \left( \text{supp } \tilde{F}_{j\xi^0 km}^I(x', x_3, y_3) \times \{(\xi^{0'}, \tau_k^-(\xi^{0'}), -\xi_3^0)\} \right) \right]$$

and

$$\text{supp } \tilde{F}_{j\xi^0 k}^I(x, y) = (\Gamma_{j\xi^0})_k^I \equiv \left\{ (x, y_3) \in \mathbf{R}_-^4 \times \mathbf{R}_- : \right. \\ \left. (x' + x_3 \text{grad}_\xi \tau_k^-(\xi^{0'})) \cdot \eta' - y_3 \eta_3 \geq 0 \text{ for any } \eta \in \Gamma_{j\xi^0} \right\},$$

$$(j, k) = \{(p_1, p_1), (p_1, s_1), (s_1, p_1), (s_1, s_1)\}$$

for  $\xi^0$  satisfying  $\tilde{F}_{j\xi^0 k}^I(x, y_3) \neq 0$ ,

$$\text{supp } \tilde{F}_{j\xi^0 km}^I(x, y) = (\Gamma_{j\xi^0 m})_k^I \equiv \left\{ (x, y_3) \in \mathbf{R}_-^4 \times \mathbf{R}_- : \right. \\ \left. (x' + x_3 \text{grad}_\xi \tau_k^-(\xi^{0'})) \cdot \eta' - y_3 \eta_3 \geq 0 \text{ for any } \eta \in \Gamma_{j\xi^0 m} \right\},$$

$$(j, k, m) = \{(p_1, p_1, p_2), (p_1, p_1, s_2), (p_1, s_1, p_2), (p_1, s_1, s_2), (s_1, p_1, p_2),$$

$$(s_1, p_1, s_2), (s_1, s_1, p_2), (s_1, s_1, s_2), (s_1, s_1, p_1)\}$$

for  $\xi^0$  satisfying  $\tilde{F}_{j\xi^0 km}^I(x, y_3) \neq 0$ .

(2) For the refracted Riemann function  $\tilde{F}^{II}(x, y_3)$ , we have

$$(3.3) \quad \lim_{\nu \rightarrow \infty} \nu e^{-i\nu\{x' \cdot \xi^{0'} + x_3 \tau_k^+(\xi^{0'}) - y_3 \xi_3^0\}} \tilde{F}^{II}(x, y_3) = \tilde{F}_{j\xi^0 k}^{II}(x, y_3),$$



$$(j, k) = \{(p_1, p_2), (p_1, s_2), (s_1, p_2), (s_1, s_2)\}$$

and if  $\xi^{0'}$  are zeros of  $\tau_m^+(\zeta')$  ( $m \in \{p_2\}$ ), then we have

$$(3.4) \quad \lim_{\nu \rightarrow \infty} \left\{ \nu^{\frac{3}{2}} e^{-i\nu \{x' \cdot \xi^{0'} + x_3 \tau_k^+(\xi^{0'}) - y_3 \xi_3^0\}} \tilde{F}^{II}(x, y_3) - \nu^{\frac{1}{2}} \tilde{F}_{j\xi^0 k}^{II}(x, y_3) \right\} = \tilde{F}_{j\xi^0 km}^{II}(x, y_3),$$

$$(j, k, m) = \{(p_1, s_2, p_2), (s_1, s_2, p_2)\}$$

in the distribution sense with respect to  $(x, y_3) \in \mathbf{R}_+^4 \times \mathbf{R}_-$ .

Moreover we have

$$WF(\tilde{F}^{II}(x', x_3, y_3)) = WF_A(\tilde{F}^{II}(x', x_3, y_3)) = \bigcup_{\xi^0 \neq 0} \left[ \bigcup_{\substack{(j,k)=\{(p_1,p_2),(p_1,s_2), \\ (s_1,p_2),(s_1,s_2)\}}} \left( \text{supp } \tilde{F}_{j\xi^0 k}^{II}(x', x_3, y_3) \times \{(\xi^{0'}, \tau_k^+(\xi^{0'}), -\xi_3^0)\} \right) \right. \\ \left. \bigcup_{(j,k,m)=\{(p_1,s_2,p_2),(s_1,s_2,p_2)\}} \left( \text{supp } \tilde{F}_{j\xi^0 km}^{II}(x', x_3, y_3) \times \{(\xi^{0'}, \tau_k^+(\xi^{0'}), -\xi_3^0)\} \right) \right]$$

and

$$\text{supp } \tilde{F}_{j\xi^0 k}^{II}(x, y_3) = (\Gamma_{j\xi^0})_k^{II} \equiv \left\{ (x, y_3) \in \mathbf{R}_+^4 \times \mathbf{R}_- : \right. \\ \left. (x' + x_3 \text{grad}_\xi \tau_k^+(\xi^{0'})) \cdot \eta' - y_3 \eta_3 \geq 0 \text{ for any } \eta \in \Gamma_{j\xi^0} \right\},$$

$$(j, k) = \{(p_1, p_2), (p_1, s_2), (s_1, p_2), (s_1, s_2)\}$$

for  $\xi^0$  satisfying  $\tilde{F}_{j\xi^0 k}^{II}(x, y_3) \neq 0$ ,

$$\text{supp } \tilde{F}_{j\xi^0 km}^{II}(x, y_3) = (\Gamma_{j\xi^0 m})_k^{II} \equiv \left\{ (x, y_3) \in \mathbf{R}_+^4 \times \mathbf{R}_- : \right. \\ \left. (x' + x_3 \text{grad}_\xi \tau_k^+(\xi^{0'})) \cdot \eta' - y_3 \eta_3 \geq 0 \text{ for any } \eta \in \Gamma_{j\xi^0 m} \right\},$$

$$(j, k, m) = \{(p_1, s_2, p_2), (s_1, s_2, p_2)\}$$

for  $\xi^0$  satisfying  $\tilde{F}_{j\xi^0 km}^{II}(x, y_3) \neq 0$ .

Here  $F_{j\xi^0 k}^I(x, y_3)$ ,  $F_{j\xi^0 km}^I(x, y_3)$ ,  $F_{j\xi^0 k}^{II}(x, y_3)$ , and  $F_{j\xi^0 km}^{II}(x, y_3)$  are localizations in the sense of (3.1), (3.2), (3.3), and (3.4), respectively, and more precise expressions are given in ([7],[8]). Moreover

$$\Gamma_{j\xi^0} = \Gamma((\det P_j^I)_{\xi^0}(\eta), \vartheta), \quad \vartheta = (1, 0, 0, 0), \quad j \in \{p_1, s_1\},$$

$$(\det P_j^I)_{\xi^0}(\eta), \vartheta) \cap \left\{ \Gamma \left( \frac{\xi_0^0}{c_m^2} \eta_0 - \xi_1^0 \eta_1 - \xi_2^0 \eta_2, \vartheta' \right) \times \mathbf{R}_\eta \right\},$$

$$\vartheta' = (1, 0, 0), \quad j \in \{p_1, s_1\}, \quad m \in \{p_2\},$$

$$\tau_{p_1}^{\pm}(\xi') = \operatorname{sgn}(\mp \xi_0) \sqrt{\frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0''' }|^2}, \quad \text{if } \frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0''' }|^2 \geq 0,$$

and  $\tau_{p_1}^{\pm}(\xi')$  is taken a branch of  $\sqrt{\frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0''' }|^2}$  such that  $\pm \operatorname{Im} \tau_{p_1}^{\pm}(\xi') > 0$  if  $\frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0''' }|^2 < 0$ .  $\tau_{s_1}^{\pm}(\xi')$ ,  $\tau_{p_2}^{\pm}(\xi')$ , and  $\tau_{s_2}^{\pm}(\xi')$  are defined as the same as  $\tau_{p_1}^{\pm}(\xi')$  substituting  $c_{p_1}$  for  $c_{s_1}$ ,  $c_{p_2}$ , and  $c_{s_2}$ , respectively.

## REFERENCES

1. M. F. Atiyah, R. Bott, L. Gårding, *Lacunae for hyperbolic differential operators with constant coefficients I*, Acta Math. **124** (1970), 109–189.
2. L. Hörmander, *On the singularities of solutions of partial differential equations*, International Conference of Functional Analysis and Related Topics, Tokyo (1969).
3. L. Hörmander, *Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients*, Comm. Pure Appl. Math. **24** (1971), 671–703.
4. L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.
5. M. Matsumura, *Localization theorem in hyperbolic mixed problems*, Proc. Japan. Acad. **47** (1971), 115–119.
6. S. Shimizu, *Eigenfunction expansions for elastic wave propagation problems in stratified media  $\mathbb{R}^3$* , Tsukuba J. Math. **18** (1994), 283–350.
7. S. Shimizu, *Singularities of solutions to elastic wave propagation problems in stratified media II*, RIMS kokyuroku **1045** (1998), 73–102.
8. S. Shimizu, *Singularities of the reflected and refracted Riemann functions of elastic wave propagation problems in stratified media*, ISAAC'97 proceeding on wave and related topics, Kluwer Academic Publishers, 1999 (to appear).
9. M. Tsuji, *Propagation of the singularities for hyperbolic equations with constant coefficients*, Japan J. Math. **2** (1976), 369–373.
10. S. Wakabayashi, *Singularities of the Riemann functions of hyperbolic mixed problems in a quarter-space*, Proc. Japan. Acad. **50** (1974), 821–825.
11. S. Wakabayashi, *Singularities of the Riemann functions of hyperbolic mixed problems in a quarter-space*, Publ. RIMS Kyoto Univ. **11** (1976), 417–440.
12. S. Wakabayashi, *Analytic wave front sets of the Riemann functions of hyperbolic mixed problems in a quarter-space*, Publ. RIMS Kyoto Univ. **11** (1976), 785–807.
13. S. Wakabayashi, *Propagation of singularities of the fundamental solutions of hyperbolic mixed problems*, Publ. RIMS Kyoto Univ. **15** (1979), 653–678.